APPM 1345
Exam 3
Spring 2024

Name	
Instructor Richard McNamara	**Section** 150

1. (23 pts) Parts (a) and (b) are unrelated.

   (a) Find the inverse function of $f(x) = \dfrac{\ln(2x)}{1 + \ln(2x)}$ for $x \geq \frac{1}{2}$.

   Express your answer in the form $f^{-1}(x)$. (You do not have to identify the inverse function's domain.)

   Solution:

$$y = \frac{\ln(2x)}{1 + \ln(2x)}$$

$$y[1 + \ln(2x)] = \ln(2x)$$

$$y + y\ln(2x) = \ln(2x)$$

$$(y - 1)\ln(2x) = -y$$

$$\ln(2x) = \frac{y}{1 - y}$$

$$2x = e^{y/(1-y)}$$

$$x = \frac{1}{2}e^{y/(1-y)}$$

Reverse the roles of x and y to get $\boxed{y = f^{-1}(x) = \frac{1}{2}e^{x/(1-x)}}$

(b) Consider the function $g(x) = 2x - \cos x$.

i. Explain why g is invertible, based on its derivative.

ii. Find an equation of the line that is tangent to the curve $y = g^{-1}(x)$ at the point $(4\pi - 1, 2\pi)$.
Hint: Do not attempt to identify the function $g^{-1}(x)$.

Solution:

i. $g'(x) = 2 + \sin x$, which is positive for all real numbers x since $-1 \le \sin x \le 1$.

Therefore, $g(x)$ is a monotone increasing function, which implies that it is invertible.

ii. The slope of the line that is tangent to the curve $y = g^{-1}(x)$ at the point $(4\pi - 1, 2\pi)$ is $(g^{-1})'(4\pi - 1)$.

Since $(g^{-1})'(x) = \dfrac{1}{g'(g^{-1}(x))}$, we know that $(g^{-1})'(4\pi - 1) = \dfrac{1}{g'(g^{-1}(4\pi - 1))}$.

Since the curve $y = g^{-1}(x)$ passes through the point $(4\pi - 1, 2\pi)$, we know that $g^{-1}(4\pi - 1) = 2\pi$.

It follows that $(g^{-1})'(4\pi - 1) = \dfrac{1}{g'(2\pi)}$.

The expression for $g'(x)$ from part (i) implies that $g'(2\pi) = 2 + \sin(2\pi) = 2$. Therefore,

$$(g^{-1})'(4\pi - 1) = \frac{1}{g'(2\pi)} = \frac{1}{2}.$$

Since the tangent line passes through the point $(4\pi - 1, 2\pi)$ its equation is

$$\boxed{y - 2\pi = \frac{1}{2}(x - (4\pi - 1))}$$

2. (25 pts) Parts (a) and (b) are unrelated.

(a)

(b) Consider the function $p(t) = p_0 e^{kt}$, which represents an exponential growth model for a population, where the constant p_0 represents the initial population size and the constant k represents the population's relative growth rate. Suppose $p(10) = 2$ and $p(50) = 6$.

i. Find the value of k.
ii. Find the value of p_0.

Solution:

The two given data points lead to the following system of two equations and two unknowns:

$$(t; p) = (10;$$

3. (26 pts) Evaluate the following derivatives using properties of logarithms and/or logarithmic differentiation. Do **not** fully simplify your answers, although they must be expressed as functions of x.

(a) $\dfrac{d}{dx}\left[\ln\left(\dfrac{(10-\cos^2 x)\sqrt{x^4+6}}{e^{x\sin x}}\right)\right]$

Solution:

$$\frac{d}{dx}\left[\ln\left(\frac{(10-\cos^2 x)\sqrt{x^4+6}}{e^{x\sin x}}\right)\right] = \frac{d}{dx}\left[\ln(10-\cos^2 x) + \ln\left[(x^4+6)^{1/2}\right] - \ln\left[e^{x\sin x}\right]\right]$$

$$= \frac{d}{dx}\left(\ln(10-\cos^2 x) + \frac{1}{2}\ln(x^4+6) - x\sin x\right)$$

$$= \frac{(-2\cos x)(-\sin x)}{10-\cos^2 x} + \frac{1}{2}\cdot\frac{4x^3}{x^4+6} - (x\cos x + \sin x)$$

$$= \boxed{\frac{2\cos x\sin x}{10-\cos^2 x} + \frac{2x^3}{x^4+6} - x\cos x - \sin x}$$

(b) $\dfrac{d}{dx}\left(e^x + e^{-x}\right)^x$

Solution:

Let $y = (e^x + e^{-x})^x$.

$$\begin{aligned}\ln y &= \ln\left(e^x + e^{-x}\right)^x \\ &= x\ln\left(e^x + e^{-x}\right)\end{aligned}$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[x\ln\left(e^x + e^{-x}\right)\right]$$

$$\frac{1}{y}\,\frac{dy}{dx} = x\,\frac{e^x - e^{-x}}{e^x + e^{-x}} + \ln\left(e^x + e^{-x}\right)$$

$$\frac{dy}{dx} = y\left[x\,\frac{e^x - e^{-x}}{e^x + e^{-x}} + \ln\left(e^x + e^{-x}\right)\right]$$

$$\frac{dy}{dx} = \boxed{\left(e^x + e^{-x}\right)^x\left[x\,\frac{e^x - e^{-x}}{e^x + e^{-x}} + \ln\left(e^x + e^{-x}\right)\right]}$$

4. (26 pts) Evaluate the following integrals.

(a) $\displaystyle\int_1^2 \frac{2^x}{9-2^x}\,dx$

Solution:

Let $u = 9 - 2^x$, which implies that $du = -2^x \ln 2\, dx$.

$x = 1 \Rightarrow u = 9 - 2^1 = 7$

$x = 2 \Rightarrow u = 9 - 2^2 = 5$

$$\int_1^2 \frac{2^x}{9-2^x}\,dx = -\frac{1}{\ln 2}\int_7^5 \frac{du}{u} = \frac{1}{\ln 2}\int_5^7 \frac{du}{u} = \boxed{\frac{\ln 7 - \ln 5}{\ln 2}}$$

(b) $\displaystyle\int \frac{x}{x-1}\,dx$

Solution:

Let $u = x - 1$, which implies that $du = dx$ and $x = u + 1$.

$$\int \frac{x}{x-1}\,dx = \int \frac{u+1}{u}\,du = \int du + \int \frac{du}{u} = u + \ln|u| + C = \boxed{x - 1 + \ln|x-1| + C}$$

Your Initials _________

ADDITIONAL BLANK SPACE

If you write a solution here, please clearly indicate the problem number.