

SOLUTION:

(a) There are three vectors in \mathbb{R}^3 , a vector space of dimension 3. Check for linear independence:

$$c_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 \cdot \frac{1}{2}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The vectors are linearly dependent and thus cannot form a basis for \mathbb{R}^3 .

(b) i. W is not a subspace. W is closed under vector addition (the sum of two integers is an integer) but is not closed under scalar multiplication. For example,

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \in W \quad \text{but} \quad \frac{1}{2} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} \notin W$$

ii. W is a subspace. Use linear combinations to check for closure. Let $u, v \in W$ and

$$u = \begin{pmatrix} 2 \\ 4 \\ u_3 \end{pmatrix} \in W \Rightarrow 4u_1 - 3u_2 + 9u_3 = 0 \quad \text{and} \quad v = \begin{pmatrix} 2 \\ 4 \\ v_3 \end{pmatrix} \in W \Rightarrow 4v_1 - 3v_2 + 9v_3 = 0$$

$$\text{Then } u + v = \begin{pmatrix} 2 \\ 4 \\ u_3 + v_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 4 \\ u_3 + v_3 \end{pmatrix} \in W$$

$$\begin{aligned} & 4(u_1 + v_1) - 3(u_2 + v_2) + 9(u_3 + v_3) \\ &= (4u_1 - 3u_2 + 9u_3) + (4v_1 - 3v_2 + 9v_3) \\ &= 0 + 0 \\ &= 0 + 0 = 0 \Rightarrow u + v \in W \end{aligned}$$

W is closed under vector addition and scalar multiplication and therefore a subspace. Note also that the equation is linear and homogeneous and hence the Superposition Principle applies, which is essentially the closure properties.

3. [2360/101922 (12 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

- (a) For any matrix A , $A^T A$ and AA^T are defined.
- (b) For invertible matrices A and B , $(AB)^{-1} = B^{-1}A^{-1}$.
- (c) If square matrix A has 0 for an eigenvalue, then Cramer's Rule can be used to solve the system $Ax = b$.
- (d) $\text{Tr}((2I)^3) = 24$ where I is the 3×3 identity matrix.
- (e) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ spans $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- (f) Homogeneous systems of linear algebraic equations are never inconsistent.

SOLUTION:

(a) **TRUE** If A is $m \times n$, then $A^T A$ is $n \times n$ and AA^T is $m \times m$ which are both square and the determinant is defined.

(b) **TRUE** $(AB)^{-1} = B^{-1}A^{-1}$

(c) **FALSE** If 0 is an eigenvalue of a square matrix, then $\det A = 0$ and Cramer's Rule cannot be used.

(d) **TRUE** $\text{Tr}((2I)^3) = \text{Tr} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \text{Tr} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 24$

(e) **FALSE** $c_1 \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has no solution. $0c_1 + 0c_2$ can never equal 3.

A basis for E_7 is $\left\{ \begin{pmatrix} 8 \\ 6 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 15 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 5 \\ 4 \end{pmatrix} \right\}$ which has dimension 3.