1. [2360/030922 (10 pts)] Given the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 4 \end{bmatrix}$$

write the word TRUE or FALSE as appropriate. No work need be shown, no work will be graded and no partial credit will be given.

(a) $CB = \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$ (b) $\mathrm{Tr}\left(B^T A^T\right) = 2$ (c) $A^T A = AA^T$ (d) $|C^T C \quad 3I| = \quad 10$ (e) $AB \quad A^T B^T$ is not defined

SOLUTION:

(a) FALSE $CB = \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 11 \end{bmatrix}$

(b) TRUE $\mathrm{Tr}\left(B^T A^T\right) = \mathrm{Tr}\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 2 \end{bmatrix} \right) = \mathrm{Tr} \begin{bmatrix} 2 & 6 & 2 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = 2 + 1 \quad 1 = 2$

(c) FALSE $A^T A$ is $(2 \quad 3)(3 \quad 2) = 2 \quad 2$ whereas AA^T is $(3 \quad 2)(2 \quad 3) = 3 \quad 3$ so they cannot be equal

(d) FALSE $|C^T C \quad 3I| = \left| \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} \quad 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix} \right|$

We need to nd constants $c_1; c_2; c_3$ such that c_1 #

(b)

$$A^TA\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\left(A^T\right)^{-1}A^TA\vec{x} = \left(A^T\right)^{-1}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{Note: } \left(A^T\right)^{-1}A^T = I \text{ and } IA = A$$

$$A\vec{x} = \left(A^T\right)^{-1}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$A^{-1}A\vec{x} = A^{-1}\left(A^T\right)^{-1}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{Note: } A^{-1}A = I \text{ and } I\vec{x} = \vec{x}$$

$$\vec{x} = A^{-1}\left(A^{-1}\right)^T\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{Note: } \left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

$$\vec{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 18 \end{bmatrix}$$

5. [2360/030922 (12 pts)] Determine if each of the following sets of vectors forms a basis for $\mathbb{R}^3$. Justify your answers.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \right\}$

SOLUTION:

Note that the dimension of $\mathbb{R}^3$ is 3 so a basis consists of 3 linearly independent vectors.

(a) The set contains only 2 vectors and thus cannot form a basis for $\mathbb{R}^3$ regardless of the linear dependence or independence of the vectors in the set.

(b) Three vectors in $\mathbb{R}^3$ can potentially be a basis if they are linearly independent. To check for this, we need to see if the only solution to

$$c_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 8 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is the trivial solution. The determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & 8 \\ 0 & 1 & 2 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 1 & 8 \\ 1 & 2 \end{vmatrix} + 2(-1)^{2+1}\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = 1(1)(-6) + 2(-1)(-3) = 0$$

implying that the system has nontrivial solutions, further implying that the vectors are linearly dependent and thus cannot form a basis for $\mathbb{R}^3$.

6. [2360/030922 (24 pts)] The following parts are unrelated.

(a) (12 pts) Find the RREF of $A = \begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & 1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix}$.

(b) (12 pts) We need to solve the system $A\vec{x} = \vec{b}$. After a number of elementary row operations, the augmented matrix for the system is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 5 \\ 0 & 1 & 3 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

i. (10 pts) Use this and the Nonhomogeneous Principle to find the solution to the original system.

ii. (2 pts) Find the dimension of the solution space of the original associated homogeneous system, $A\vec{x} = \vec{0}$. Hint: You have the information you need from part (i); very little additional work is required.

SOLUTION:

(a)
$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & 1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix} \begin{array}{l} R_2 = 1R_1 + R_2 \\ R_3 = 3R_1 + R_3 \end{array} \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & 2 & 2 & 8 \\ 0 & 2 & 2 & 8 \end{bmatrix} \begin{array}{l} R_3 = R_2 + R_3 \\ R_2 = \frac{1}{2}R_2 \end{array} \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 = 3R_2 + R_1 \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) i. Pivot columns correspond to x_1, x_2, x_4 so these are basic variables with x_3 and x_5, corresponding to the nonpivot columns, being free variables. Setting $x_3 = s$ and $x_5 = t$, solutions have the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \quad 3t \\ 4 \quad 3s + 2t \\ s \\ 1 + 2t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad \text{where } s, t \in \mathbb{R}$$

$$= \vec{x}_p + \vec{x}_h$$

ii. A basis for the solution space of the associated homogeneous system is $\left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$, containing two linearly independent vectors so its dimension is 2.

7. [2360/030922 (14 pts)] Determine if the subsets, W, are subspaces of the given vector spaces, V.

(a) (7 pts) $V = M_{22}$; $W = \left\{ A \in M_{22} \;\middle|\; A^T = A \right\}$, the set of all matrices of the form $\begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$ where k is a real number.

(b) (7 pts) $V = \mathbb{R}^3$; $W = \left\{ \vec{v} \in \mathbb{R}^3 \;\middle|\; \vec{v} = \begin{bmatrix} p + q \\ r \\ s \end{bmatrix} \text{ where } p, q, r, s \in \mathbb{R} \text{ and } s \quad 0 \right\}$;

SOLUTION:

(a) Clearly $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$. Let $\vec{u} = \begin{bmatrix} 0 & u \\ u & 0 \end{bmatrix} \in W$ and $\vec{v} = \begin{bmatrix} 0 & v \\ v & 0 \end{bmatrix} \in W$ and $p, q \in \mathbb{R}$. Then

$$p\vec{u} + q\vec{v} = p\begin{bmatrix} 0 & u \\ u & 0 \end{bmatrix} + q\begin{bmatrix} 0 & v \\ v & 0 \end{bmatrix} = \begin{bmatrix} 0 & pu \\ pu & 0 \end{bmatrix} + \begin{bmatrix} 0 & qv \\ qv & 0 \end{bmatrix} = \begin{bmatrix} 0 & pu + qv \\ pu \quad qv & 0 \end{bmatrix} = \begin{bmatrix} 0 & pu + qv \\ (pu + qv) & 0 \end{bmatrix} \in W$$

since

$$\begin{bmatrix} 0 & pu + qv \\ (pu + qv) & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & (pu + qv) \\ pu + qv & 0 \end{bmatrix} = \begin{bmatrix} 0 & pu + qv \\ (pu + qv) & 0 \end{bmatrix}$$

The set is closed under linear combinations and thus is a subspace.

Alternatively, let $A, B \in W$ and $\;,\; \in \mathbb{R}$. Let $C = A + B$: Then

$$C^T = (A + B)^T = A^T + B^T = A \quad B = (A + B) = C:$$

Therefore $C \in W$, so by the Vector Subspace Theorem, W is a subspace of V.

(b) Let $\vec{v} = \begin{bmatrix} p+q \\ r \\ s \end{bmatrix} \in W$ with $s > 0$. Then $-1\vec{v} = \begin{bmatrix} -p-q \\ -r \\ -s \end{bmatrix} \notin W$ since $-s < 0$. This implies that W is not closed under scalar multiplication and thus is not a subspace.