1. [2360/072222 (30 pts)] Solve the initial value problem

(j) **TRUE**. The change of variable $y = 50 - T$ yields $y' = -2y$.

3. (35 pts) The following parts are not related.

   (a) (10 pts) Consider the initial value problem (IVP) $\quad ty' - 3(\ln t)^2 e^{-y} = 0; \quad y(1) = \ln 8.$

      i. (8 pts) Find the implicit solution to the IVP.

      ii. (2 pts) Find the explicit solution to the IVP and state the interval over which the solution is valid.

   (b) (25 pts) A particular solution to $L(\vec{y}) = f$, where L is a linear operator, is $y_p = \cos t$. Suppose the characteristic equation for the associated homogeneous equation is $(r-2)(r^2-1) = 0$. Use Cramer's Rule to find the solution to the following initial value problem. No points for using other methods.

$$L(\vec{y}) = f; \quad y(0) = 4;\ y'(0) = 0;\ y''(0) = -1$$

SOLUTION:

(a) i. The equation is separable.

$$\int e^y\,dy = \int 3\frac{(\ln t)^2}{t}\,dt \quad (u = \ln t)$$
$$e^y = 3\int u^2\,du = (\ln t)^3 + C$$
$$e^{\ln 8} = (\ln 1)^3 + C \implies C = 8$$
$$e^y = (\ln t)^3 + 8$$

ii. The explicit solution is $y = \ln\left[(\ln t)^3 + 8\right]$. Clearly, $t > 0$ for input into the "inner" $\ln t$. For input into the "outer" natural logarithm function, we also need

$$(\ln t)^3 + 8 > 0 \implies \ln t > \sqrt[3]{-8} = -2 \implies t > e^{-2}$$

The solution is valid on $\left(e^{-2}, \infty\right)$.

(b) Based on the characteristic equation, $(r-2)(r+1)(r-1) = 0$, the solution to the homogeneous equation is $y_h = c_1e^{2t} + c_2e^{-t} + c_3e^{t}$ so the general solution to which we apply the initial conditions is $y = y_h + y_p$.

$$y(t) = c_1e^{2t} + c_2e^{-t} + c_3e^{t} + \cos t$$
$$y'(t) = 2c_1e^{2t} - c_2e^{-t} + c_3e^{t} - \sin t$$
$$y''(t) = 4c_1e^{2t} + c_2e^{-t} + c_3e^{t} - \cos t$$

At $t = 0$ we have

$$\begin{aligned} c_1 + c_2 + c_3 + 1 &= 4 \\ 2c_1 - c_2 + c_3 &= 0 \\ 4c_1 + c_2 + c_3 - 1 &= -1 \end{aligned} \implies \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Now use Cramer's Rule

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} = 1(-1)$$

The solution to the initial value problem is thus

$$y(t) = \quad e^{2t} + e^{\quad t} + 3e^{t} + \cos t$$

4. [2360/072222 (29 pts)] The following parts are not related.

(a) (12 pts) Consider the function $f(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \quad t < 2 \\ 5 \quad t & 2 \quad t \end{cases}$

i. (3 pts)

Thus

$$\mathcal{L}\left\{t^2\,\mathrm{step}(t) - t^2\,\mathrm{step}(t-2) + (5-t)\,\mathrm{step}(t-2)\right\} = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) + e^{-2s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{5}{s^2} + \frac{1}{s}\right)$$

(b) i. Let $x_1(t); x_2(t); x_3(t)$ represent the mass (grams) of sugar and $V_1(t); V_2(t); V_3(t)$ the volume of solution (L) in Tank 1, 2, 3 at time t, respectively. Then with

$$\frac{dV}{dt} = \text{flow rate in} - \text{flow rate out}$$

we have

$$\frac{dV_1}{dt} = 1 + 4 - 6 = -1 \quad V_1(0) = 100 \implies \int dV_1 = \int$$

(a)

$$\begin{matrix}1 & & 1\\ & 9 & & 5\end{matrix} = (\ 1\ )(5\ ) + 9 = {}^{2}\ 4\ + 4 = (\ 2)^2 = 0 \implies\ = 2 \text{ with multiplicity 2}$$

We need to find nontrivial solutions to $(\mathrm{A}\ \ 2\mathrm{I})\vec{v} = \vec{0}$ giving

$$\left[\begin{array}{cc|c} 3 & 1 & 0\\ 9 & 3 & 0\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0\\ 0 & 0 & 0\end{array}\right] \implies v_1 = \frac{1}{3}v_2 \implies \vec{v} = \begin{bmatrix}1\\3\end{bmatrix}$$

Since there is only one eigenvector, we need to find the generalized eigenvector by finding a nontrivial solution to $(\mathrm{A}\ \ 2\mathrm{I})\vec{u} = \vec{v}$.

$$\left[\begin{array}{cc|c} 3 & 1 & 1\\ 9 & 3 & 3\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3}\\ 0 & 0 & 0\end{array}\right] \implies u_1 = \ \frac{1}{3} + \frac{1}{3}u_2 \implies \vec{u} = \begin{bmatrix}1\\4\end{bmatrix}$$

The general solution is

$$\vec{x}(t) = c_1e^{2t}\begin{bmatrix}1\\3\end{bmatrix} + c_2e^{2t}\left(t\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1\\4\end{bmatrix}\right)$$

Applying the initial condition yields

$$c_1\begin{bmatrix}1\\3\end{bmatrix} + c_2\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

with Cramer's Rule giving

$$c_1 = \frac{\begin{vmatrix}2 & 1\\ 1 & 4\end{vmatrix}}{\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}} = \frac{9}{1} = 9 \qquad c_2 = \frac{\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix}}{\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}} = \frac{\ 7}{1} = \ \ 7$$

and the final solution to the initial value problem as

$$\vec{x} = e^{2t}\begin{bmatrix}2 & 7t\\ 1 & 21t\end{bmatrix}$$

(b) We have $\mathrm{Tr}\,\mathrm{A} = 2k; |\mathrm{A}| = k^2 + 2$ and $(\mathrm{Tr}\,\mathrm{A})^2 \ \ 4|\mathrm{A}| = 4k^2 \ \ 4(k^2 + 2) = \ \ 8$

i. **All real numbers** Since $|\mathrm{A}| = k^2 + 2 \neq 0$ for all k, the system $\mathrm{A}\vec{x} = \vec{0}$ has only the trivial solution for all values of k. Thus, regardless of the value of k, the system will always have a unique equilibrium solution at $(0, 0)$.

ii. **None** $|\mathrm{A}| = k^2 + 2 > 0$ 4

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