Write your name and your professor's name or your section number in the top right corner of your paper. You are allowed to use textbooks and notes, but you may not ask anyone for help except the professors. To receive full credit on a problem you must show **su cient justi cation for your**

3. (20 points) Let

$$L(x, y, z) = \begin{pmatrix} 2(y - z) + x \\ -4z + 2y \\ -z \end{pmatrix}.$$

Find the matrix representation of L with respect to the following basis of $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Solution: First find the matrix representation with respect to the standard basis by plugging in the standard basis vectors and placing the output as columns of

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}.$$

Next arrange the new basis vectors as columns of a matrix

$$S = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The solution is found using $S^{-1}AS = \begin{bmatrix} -1 & 6 & 2 \\ -1 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}$. Although you don't actually need S^{-1} to find the solution, here it is for reference, along with AS:

$$S^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix}; \quad AS = \begin{pmatrix} -1 & 6 & 0 \\ -2 & 6 & -4 \\ 0 & 0 & -1 \end{pmatrix}$$

4. (20 points) Suppose that you have the following data from 100 people: weight w_i, height h_i, age t_i and blood pressure p_i (where $i = 1, \dots, 100$). You decide to model the person's blood pressure p as a function of the other factors as follows $p = \beta_0 + \beta_1 \frac{w}{h^2} + \beta_2 t + \beta_3 t^2$. What are the entries of the matrix A and vector $\vec{b}$ such that the least-squares solution of $A\vec{x} = \vec{b}$ is the vector of linear regression coefficients $\beta_0, \dots, \beta_3$?

The linear system is obtained by plugging the data into the model, which yields the system

$$\begin{aligned} \beta_0 + \beta_1 \frac{w_1}{h_1^2} + \beta_2 t_1 + \beta_3 t_1^2 &= p_1 \\ &\vdots \\ \beta_0 + \beta_1 \frac{w_{100}}{h_{100}^2} + \beta_2 t_{100} + \beta_3 t_{100}^2 &= p_{100} \end{aligned}$$

This linear system can be written in matrix form $Ax = b$ with

$$A = \begin{bmatrix} 1 & \frac{w_1}{h_1^2} & t_1 & t_1^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{w_{100}}{h_{100}^2} & t_{100} & t_{100}^2 \end{bmatrix}; \quad x = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}; \quad b = \begin{pmatrix} p_1 \\ \vdots \\ p_{100} \end{pmatrix}.$$

5. (24 points) Find the singular value decomposition of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Solution: The Gram matrix is

$$A^T A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

The eigenvalues of the Gram matrix are 0 and 2, so there is only one singular value $\sigma_1 = \sqrt{2}$. The associated singular vector is $(1, 1)^T$, which has to be normalized by dividing by $\sqrt{2}$. There is only one p_i vector, found using $p_1 = \sigma_1^{-1} A q_1$. The final answer is

$$P = \frac{1}{\sqrt{2}}$$