3. Let $N \geq 0$ be an unknown integer and let $X_1, \ldots, X_n$ be a random sample from the distribution with

$$P(X_i = k) = \begin{cases} \frac{1}{2N+1} & ; \ k = 0, \pm 1, \pm 2, \ldots, \pm N; \\ 0 & ; \ \text{otherwise}. \end{cases}$$

(a) Show that $M = \max\{|X_1|, \ldots, |X_n|\}$ is a sufficient statistic for N.

(b) Show that M is a complete statistic for N.

(c) Determine the uniformly minimum variance unbiased estimator (UMVUE) for N. Simplify your answer!
Hint for part (c). Determine constants a, b, c such that $a\,|X_1| + b\,[\![X_1 = 0]\!] + c$ is unbiased for N, where $[\![X_1 = 0]\!]$ is notation for the indicator function of the event $[X_1 = 0]$.

4. Suppose that X_1 is a random sample of size 1 from the distribution with probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{2\theta} & ; \ \text{if } 0 < x \leq \theta; \\ \frac{\theta}{2x^2} & ; \ \text{if } x \geq \theta; \\ 0 & ; \ \text{otherwise}. \end{cases}$$

For which values of $0 < \alpha < 1$ does $H_0 : \theta = 1$ versus $H_1 : \theta \geq 3$ admit a unique uniformly most powerful (UMP) test of size α? Specify the rejection region associated with each of those tests.

5. Let $0 < \lambda < \mu$ be real constants and consider an M/M/1 queueing system with arrival rate λ and service rate μ, which is initially empty. Suppose that the server is turned ON and OFF according to the following rules:

- it remains OFF as long as the number of customers in the system is less than 2,
- it is turned ON as soon as the number of customers in the system becomes 2 and then remains ON until completely emptying the system of customers.

Based on the above, please respond:

(a) Model the number of customer in the system as a continuous time Markov chain with state space $\{0, 1', 1, 2, 3, \ldots\}$, where $1'$ represents the configuration of having exactly one customer in the system while the server is OFF, and state 1 represents the same but while the server is ON. Represent the rate transition matrix of the chain as a directed graph with weighted edges.

(b) Show that this system has a stationary distribution and determine it explicitly.

(c) After a long time of operation, what is the probability that a new customer encounters the queue empty?