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Cointegrated Sectoral Productivities and Investment-Specific Technology in U.S. Business Cycles

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Abstract

Applying Johansen cointegration test to U.S. annual data constructed from the EU KLEMS database, the paper documents that the productivities of consumption-goods and equipment-goods sector are cointegrated. It con rms further, using the non-linear cointegration test framework developed by Kapetanioset al. (2006), that the cointegrating relation is non-linear. The cointegration of sectoral productivities is also documented in the empirical ndings of Schmitt-Grohe and Uribe (2011). I successfully derive a theoretical proposition that implies that sectoral productivities of the consumption-goods and equipment-goods sectors are cointegrated if and only if the aggregate neutral productivity and the investment-speci c technology are cointegrated. Plus, I consider the non-linear cointegration of sectoral productivities to examine the role of the common stochastic trend of sectoral productivities in explaining the movements of investment-speci c technology as well as those of interesting macroeconomic aggregates such as output, consumption, investment and hours worked. For this end, I construct a two-sector dynamic stochastic general equilibrium (DSGE) model where the productivities of the consumption and equipment sectors feature a non-linear error correction (NEC) in the vector error correction model (VECM). The maximum likelihood estimation successfully estimates most of structural parameters, including the sectoral capital shares, and it identi es all structural shocks. The paper nds that the innovations of common stochastic trends of sectoral productivities account for half of consumption, 79 percent of investment, and only 6 percent of hours worked variabilities in long-run.

Keywords: Two-sector model; Business cycles; Investment-speci c technology; Productivity; Cointegrateion; Non-linear error correction

JEL Classi cation Numbers: E32

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1 Introduction

Since the seminal work of Greenwood, Hercowitz and Krusell (1997, 2000), investment-speci c technology (IST) has become a leading candidate as a main source of economic growth and uctuation rather than total factor productivity (TFP). They suggest also that IST can be expressed by the ratio of the productivity in the equipment sector to that in the consumption-goods sector. There is a hardship, however, in interpreting the progress of IST as technological progress of the capital-goods (or equipment) sector.[1] Oulton (2007) suggests that IST may change without a change in the di erence of sectoral productivities between consumption-goods and equipment.[2] Furthermore, Whelan (2003) insists that a two-sector approach incorporating relatively high technological progress of durable goods better explain the long-run behavior of the U.S. economy. As another modi cation to the IST literatures, Schmitt-Grhœ and Uribe (2011) introduce a cointegrated relationship between TFP and IST, which is supported by an empirical analysis that shows a common stochastic trend in TFP and IST. They insist that the innovation in the common stochastic trend explains a sizeable fraction of volatilities of output, consumption, investment, and hours.

To investigate business cycles features in the U.S. economy, this paper considers the two ways of modi cation exhibited above. Ireland and Schuh (2008) establish a two-sector economy model incorporating both level and growth-rate shocks of sectoral productivities, inspired by Whelan (2003), to study the U.S. business cycles. Their study, however, does not re ect the fact that the sectoral productivities are cointegrated. Therefore, one key feature of this study is the cointegrated relationship between sectoral productivities.

What makes the cointegrated sectoral productivities so important in business cycles studies? Sectoral production performance is a ected by the amount of factor inputs, such as labor and capital, and sector-speci c production knowledge as well as some countrywide environments such

[1] Recent empirical studies show that the relative price of capital goods does not correctly measure the relative productivity changes. Basu et al. (2010) estimate technological changes at a disaggregated industry level and aggregate them by using the U.S. input-output tables. Their nding suggests that relative price does not properly measure the relative technological change. Adopting the two-sector model calibrated on the U.S. input-output tables, Guerrieri et al. (2010) conclude that the e ect of TFP in the machinery sector is qualitatively di erent from that of IST.conclud0.398 ludr6ncaprontur3sntur3sneoonc(on)-2(0))-5eoonc1onely dialelrg ssb9rTJ 0 -1(and)58(ds)- themcs-3er63erenc-458(b(

as infrastructure, education, politics, culture, and so on. In the neoclassical growth accounting framework, we can derive a sectoral TFP as a residual measure, called Solow residuals. In turn, the schedule of sectoral TFPs depends on sector-specific production knowledge as well as country-wide economic environments that affect the production of all sectors simultaneously. Accordingly, there may exist a common stochastic trend among sectoral TFPs, which implies the cointegrated relationship in sectoral productivities.

To shed light on the cointegrated relationship, two independent analyses are performed. First, I conduct the Johansen cointegration test on two sectoral productivities of consumption-goods and equipment sectors, which are reconstructed from the EU KLEMS database[3]. The test statistics confirm the cointegration between sectoral productivities[4]. As the second way to illuminate sectoral cointegration in productivity, I establish theoretical propositions based on the findings of Schmitt-Grohé and Uribe (2011) that the aggregate neutral productivity and IST are cointegrated. The propositions imply that the sectoral productivities are cointegrated if and only if the aggregate neutral productivity and IST are cointegrated. Thereby the sectoral cointegrated relationship is supported by the empirical findings of Schmitt-Grohé and Uribe (2011).

Applying the cointegration of sectoral productivities into a dynamic stochastic general equilibrium (DSGE) model, the present paper examines the effects and roles of each structural shock, such as the shocks of preference and productivities, in the U.S. business cycles. As in Ireland and Schuh (2008), the level and growth-rate shocks of preference, and those of the productivities of consumption-goods and equipment sectors are employed. To incorporate the cointegrated relationship of sectoral productivities into the DSGE model, we have to consider the fact that the cointegrated relationship of sectoral productivities may possess a dynamic instability, if the long-run equilibrium between the sectoral productivities is not linear. To resolve this problem and ensure globally-stationary error correction dynamics, I introduce a smooth transition non-linear error correction (STR NEC) featured by exponential function into the vector error correction model

[3] For more details about the EU KLEMS database, refer to O'Mahony and Timmer(2009). The data is available at www.euklems.net.

[4] Marquis and Trehan (2008) capture the idea that the productivities of consumption-goods and equipment shares common shocks. They fail to estimate, however, the cointegrated relationship between sectoral productivities, and just incorporate the correlation between the growth rate of the equipment productivity and that of consumption-goods productivity.

(VECM) framework for sectoral productivities. Using the established stationary model, I perform the maximum likelihood estimation to estimate the deep parameters including sectoral capital shares without symmetric assumption. The model estimation successfully identifies all parameters. The estimated sectoral capital shares confirm the conventional wisdom that consumption-goods sector is relatively labor-intensive, whereas equipment sector is capital-intensive. More importantly, different to Ireland and Schuh (2008) which fail to identify the growth rate shock of equipment sector, this paper successfully identifies all structural shocks.

As results, I find a sizeable effect of common stochastic trends in sectoral productivities to business cycles with persistence. Innovations in the common stochastic trends, which mostly rely on the equipment sector, increase consumption and investment almost permanently, and explains the long-run variabilities of about 48 percent and 79 percent in consumption and investment, respectively, and account for only 6 percent of hours-worked variability. Similarly to Ireland and Schuh (2008), the innovation of preference gives highly persistent and sizeable effects on hours-worked. Also, the preference shocks account for half of consumption variability and most of hours-worked variability. The level shocks of productivities explain only short-run fluctuations; there is no persistence in these shocks.

The remainder of the paper is organized as follows. Section 2 illuminates the cointegrated relationship in the U.S. sectoral productivities both in empirical and theoretical ways. Section 3 establishes a model economy incorporating the cointegrated sectoral productivities. Section 4 estimates the model with the maximum likelihood and discusses the estimates. Section 5 examines the impulse responses and the contributions of structural shocks to forecast error variance. Lastly, Section 6 concludes this paper.

2 Cointegrated productivities

sector j out of the total demand of sector i, which satisfies $\sum_j \omega^i_{K,j,t} = 1, \forall t$. The aggregations for sectoral output, intermediate input, and labor services adopt the same method of capital service.

normalize indices with the value of base year 1995.

Table 1: Unit-root tests for the logarithms of productivities and relative price of equipment

Data	Test	Trend	Lags (AIC)	Test-stats.	Critical values (5%)	Null hypothesis
TFP.cons	ADF	No	1	1.15	-1.95	Accept
	ADF	Yes	1	-2.12	-3.5	Accept
	DF-GLS	No	1	-0.319	-1.95	Accept
	DF-GLS	Yes	1	-2.38	-3.19	Accept
TFP.equip	ADF	No	1	2.72	-1.95	Accept
	ADF	Yes	1	-0.46	-3.5	Accept
	DF-GLS	No	1	1.48	-1.95	Accept
	DF-GLS	Yes	1	-0.976	-3.19	Accept
TFP.tot	ADF	No	1	1.83	-1.95	Accept
	ADF	Yes	1	-1.44	-3.5	Accept
	DF-GLS	No	1	0.901	-1.95	Accept
	DF-GLS	Yes	1	-1.93	-3.19	Accept
RP	ADF	No	1	-3.07	-1.95	Reject
	ADF	Yes	1	-0.357	-3.5	Accept
	DF-GLS	No	1	1.48	-1.95	Accept
	DF-GLS	Yes	1	-0.772	-3.19	Accept

Notes: All unit-root tests fail to reject except the ADF test for RP without trend. Tests are conducted using the R program with the \urca" package. ADF stands for Augmented Dickey-Fuller, and DF-GLS stands for Dickey-Fuller Generalized Least Squares. TFP.cons, TFP.equip, TFP.tot, and RP denote the productivity of consumption goods sector, the productivity of equipment sector, the productivity of aggregate economy, and the relative price of equipment, respectively.

Empirical ndings

Unit-root and cointegration tests are conducted for the logarithms of aggregated TFP, sectoral productivities, and relative price of equipment by using the data constructed above. As rst, augmented Dickey-Fuller (ADF) and Dickey-Fuller GLS (DF-GLS) tests are performed to test the unit root. Table 1 presents the results. The ADF test fails to reject the unit-root hypothesis except for the relative price of equipment without trend. DF-GLS can be considered as the increased power of the test, but it cannot reject the null hypothesis of unit root in all tested variables in both with and without trend. I also conduct the unit-root tests for the rst-di erenced logged variables, which are not reported here, and all test statistics reject the null hypothesis. Based on the results so far, I can therefore conclude that logged aggregate TFP, TFP in consumption-goods, TFP in equipment

Table 2: The Johansen trace test for cointegration

Database	Cointegration rank	Lags (AIC)	Test-stats.	Critical values (5%)	Null hypothesis
db1	r < = 2	3	0.103	8.18	-
	r < = 1		13.524	17.95	Accept
	r = 0		40.328	31.52	Reject
db2	r < = 2	3	0.35	8.18	-
	r < = 1		7.31	17.95	Accept
	r = 0		37.00	31.52	Reject
db3	r < = 2	3	0.0765	8.18	-
	r < = 1		7.4565	17.95	Accept
	r = 0		37.0785	31.52	Reject
db4	r < = 2	3	0.433	8.18	-
	r < = 1		7.375	17.95	Accept
	r = 0		36.863	31.52	Reject
db5	r < = 1	3	1.62	8.18	Accept
	r = 0		21.13	17.95	Reject
db6	r < = 1	3	0.324	8.18	Accept
	r = 0		20.898	17.95	Reject

Notes: The Johansen trace tests con rm cointegrated relation for all speci ed datasets with one cointegrating vector. Tests are conducted using the R program with the \urca" package. Test models don't include both constant and trend. The dataset used for the Johansen cointegration test are de ned as follows:
db1: TFP.tot, TFP.cons, TFP.equip
db2: RP, TFP.cons, TFP.equip
db3: TFP.tot, RP, TFP.cons
db4: TFP.tot, RP, TFP.equip
db5: TFP.tot, RP
db6: TFP.cons, TFP.equip

and relative price of equipment are integrated by order one.

Schmitt-Grohe and Uribe (2011) nd the cointegration of TFP and relative price of equipment with the U.S. quarterly data. To con rm the consistency of their result, I conduct Johansen cointegration tests with various sets of variables including the dataset of TFP and the relative price of equipment with the U.S. data from the EU KLEMS database. The test results of the Johansen trace and maximum eigenvalue tests are exhibited in Table 2 and 3, respectively.

Both Johansen tests, trace and maximum eigenvalue, con rm that the system of logged aggregate TFP and sectoral productivities (db1) have one cointegrating vector, which implies logged TFP can be expressed as a linear combination of two sectoral productivities and one anonymous stationary series. Conventional wisdom on growth accounting also supports this result. The system

Table 3: The Johansen maximum eigenvalue test for cointegration

Database	Cointegation rank	Lags (AIC)	Test-stats.	Critical values (5%)	Null hypothesis
db1	r = 2	3	0.103	8.18	-
	r = 1		13.421	14.9	Accept
	r = 0		26.804	21.07	Reject
db2	r = 2	3	0.35	8.18	-
	r = 1		6.96	14.9	Accept
	r = 0		29.68	21.07	Reject
db3	r = 2	3	0.0765	8.18	-
	r = 1		7.3799	14.9	Accept
	r = 0		29.6221	21.07	Reject
db4	r = 2	3	0.433	8.18	-
	r = 1		6.941	14.9	Accept
	r = 0		29.489	21.07	Reject
db5	r = 1	3	1.62	8.18	Accept
	r = 0		19.50	14.9	Reject
db6	r = 1	3	0.324	8.18	Accept
	r = 0		20.574	14.9	Reject

Notes: The Johansen maximum eigenvalue tests con rm the cointegrated relation for all speci ed datasets with one conintegrating vector. Tests are conducted using the R program with the \urca" package. Test models don't include both constant and trend. The dataset used for the Johansen cointegration test are de ned as follows:

db1: TFP.tot, TFP.cons, TFP.equip

db2: RP, TFP.cons, TFP.equip

db3: TFP.tot, RP, TFP.cons

db4: TFP.tot, RP, TFP.equip

db5 -10.959 Td [(db5 -10.959 Tdc50 -10.457(JRP)).equip cointegrated relation of sectoral productivities. The cointegration test for sectoral productivities (db6) con rms that the inference is right.

The cointegrated relation among sectoral productivities indicates the possibility that the co-movements of aggregate variables and sectoral comovements can arise not only from structural linkages but also from common stochastic trends. Most of the literature in multi-sector business cycles has investigated the sectoral comovements with sectoral structural linkages: Hornstein and Praschink (1997), and Horvath (2002) incorporate intermediate inputs into their model economy to foster sectoral linkages and nd positive sectoral comovement in output and employment. However, the empirical ndings in Tables 2 and 3, which exhibit the existence of a common stochastic trend in sectoral productivities, suggest that the common stochastic trend of sectoral productivities is another key to solving the sectoral comovement puzzle.

2.2 Theoretical approach

Schmitt-Grohe and Uribe (2011) exhibit that the U.S. quarterly data indicate that the neutral productivity and IST share common stochastic trends. Then, where do the stochastic trends come from? To address this question, I rst ignore the empirical results of the previous subsection except for the ndings of Schmitt-Grohe and Uribe (2011). There are two reasons. First, the cointegration test with annual data is sensitive to lag selection due to the small sample property. Hence, the ndings of quarterly data ranging 1948-2006 are much more reliable compared to the annual data. Secondly, I show that the existence of the common trends in sectoral productivity can be proven without using the sophisticatedly disaggregated high-quality database.

Since Greenwood

aggregate social utility, $U(C_t; N_t)$, in an in nite time horizon with the given resource constraint,

$$C_t + J_t = Y_t; \tag{1}$$

where C_t is an aggregate consumption, J_t is a forgone consumption or savings for investment spending, and Y_t is a composite output consisting of consumption goods and equipment. The investment spending is used for purchasing equipment and eventually contributes to capital accumulation as follows:

$$K_{t+1} = (1 \quad)K_t + I_t; \tag{2}$$

where K_t is a capital stock at the beginning of period t, implies depreciation rate of capital stocks, and I_t stands for the amount of newly produced equipment used for gross investment during period t. Note that the gross investment, I_t, is measured in the unit of equipment, whereas the investment spending, J_t, takes the unit of consumption. In capital accumulation the investment spending must be therefore transformed into the unit of equipment. Suppose that Q_t governs the linear transformation of the forgone consumption, then we can rewrite Eq.(2) as[8]

$$K_{t+1} = (1 \quad)K_t + J_t Q_t: \tag{3}$$

Since the nominal investment spending, $P_{c;t} J_t$, should equal the market value of investment, $P_{e;t} I_t$, Eq.(2) and Eq.(3) imply

$$Q_t \quad \frac{P_{c;t}}{P_{e;t}}; \tag{4}$$

where $P_{c;t}$ is the market price of consumption goods, $P_{e;t}$ is the price for newly produced equipment and Q_t is known as IST from Greenwood et al. (1997).

Each representative producer of both sectors uses capital and labor in its constant return to

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scale production function with its own neutral technological progress as follows:

$$Y_{c,t} = Z_{c,t} F^{c}(K_{c,t}, N_{c,t}), \tag{5}$$

$$Y_{e,t} = Z_{e,t} F^{e}(K_{e,t}, N_{e,t}), \tag{6}$$

where $Y_{c,t}$ and $Y_{e,t}$ are the outputs of consumption goods and equipment sector, respectively. $K_{j,t}$ and $N_{j,t}$ stand for capital and labor inputs, respectively, of sector $j \in \{c, e\}$. The sum of each input across sectors satisfies the feasibility conditions: $N_t \equiv N_{c,t} + N_{e,t}$ and $K_t \equiv K_{c,t} + K_{e,t}$. Suppose that $Z_{j,t}$ represents the neutral productivity of sector j and has a random walk process as follows:

$$\ln Z_{c,t} = \ln Z_{c,t-1} + \epsilon_{c,t}, \tag{7}$$

$$\ln Z_{e,t} = \ln Z_{e,t-1} + \epsilon_{e,t}, \tag{8}$$

where both $\epsilon_{c,t}$ and $\epsilon_{e,t}$ are independent white noises. Note that both sectoral productivities follow uncorrelated random walk processes due to the independently distributed disturbances, $\epsilon_{c,t}$ and $\epsilon_{e,t}$.

Suppose both sectors are in perfect competition, then the representative firms would set their prices at marginal costs, which implies

$$\frac{P_{c,t}}{P_{e,t}} = \frac{Z_{e,t} F^{e}_{1}(K_{e,t}, N_{e,t})}{Z_{c,t} F^{c}_{1}(K_{c,t}, N_{c,t})}, \tag{9}$$

where $F^{j}(\cdot,\cdot)$ is a constant-returns production function of sector j and $F^{j}_{1}(\cdot,\cdot)$ is the partial derivative with respect to the first argument. By considering the equivalence for IST, the inverse relative price of equipment given by Eq.(4), and the constant returns of production function, we can rewrite Eq.(9) as

$$Q_t = \frac{Z_{e,t} f^{e\prime}(k_{e,t})}{Z_{c,t} f^{c\prime}(k_{c,t})}, \tag{10}$$

where $k_{j,t}$ exhibits a capital per worker in sector j and $f^{j}(k_{j,t}) = F^{j}(K_{j,t}/N_{j,t}, 1)$. Suppose further that the production function is Cobb-Douglas such that $f^{j}(k_{j,t}) = k_{j,t}^{\alpha_j}$, then Eq.(10) is extended by logged variables as

$$\ln Q_t = \ln Z_{e;t} \quad \ln Z_{c;t} + S_{q;t}; \tag{11}$$

where $S_{q;t} = \ln \quad _e \quad \ln \quad _c \quad (1 \quad _e) \ln k_{e;t} + (1 \quad _c) \ln k_{c;t}$, and $_j$ indicates the capital share of sector j. Without loss of generality, we can assume that the capital/worker ratios of both sectors change with a deterministic trend, which implies a trend-stationary stochastic process. Thus, $S_{q;t}$ is stationary. Since logged Q_t is composed of two uncorrelated random walk processes and a stationary process, the investment-speci c productivity, Q_t, also has a random walk process.

On the other hand, the composite output Y_t consists of $Y_{c;t}$ and $Y_{e;t}$ with an aggregator $(\;)$. To make things more precise, suppose that the aggregator is Cobb-Douglas as

$$Y_t = \; (\; Y_{c;t}; Y_{e;t}) = \; Y_{c;t} \; Y_{e;t}^{1} \quad ; \tag{12}$$

where $\; 2 \; [0;1]$ indicates the share of output for consumption goods to the total output. Using the production functions given in Eq.(5) and Eq.(6), the composite output can be extended by logged variables as

$$\begin{aligned} \ln Y_t \quad = & \quad \ln Z_{c;t} + (1 \quad) \ln Z_{e;t} \\ & + \quad _c \; \ln K_{c;t} + \quad _e(1 \quad) \ln K_{e;t} \\ & + (1 \quad _c) \quad \ln N_{c;t} + (1 \quad _e)(1 \quad) \ln N_{e;t}; \end{aligned}$$

which implies that the Solow residuals of the aggregate output from a typical growth accounting method is a linear combination of $\ln Z_{c;t}$ and $\ln Z_{e;t}$:

$$\ln A_t \quad \ln Z_{c;t} + (1 \quad) \ln Z_{e;t}; \tag{13}$$

where A_t represents Solow residuals or the aggregate TFP.

Then, logged A_t has to be a random walk because logged $Z_{c;t}$ and $Z_{e;t}$ are uncorrelated $I(1)$

yields

$$\ln Q_t \quad (1 \qquad) \; ^{1} \ln A_t + (1 \qquad) \; ^{1} \ln Z_{c;t} = S_{q;t}: \qquad (14)$$

walk assumption from both sectoral productivities to either one of the two. This modi cation does not hurt the non-stationary property of the aggregate neutral and investment-speci c productivities, while ensuring cointegration between them; at least one non-stationary process is enough to make any linear combination of productivities non-stationary. However, this has not been supported by data. According to Table 1, U.S. sectoral productivities constructed from the EU KLEMS database reveal that the sectoral productivities have $I(1)$ processes in both sectors.

Another possible modi cation is introducing a cointegrated relation of both sectoral productivities, which is also supported by the empirical results for \db6" in Tables 2 and 3. To derive a formal theoretical result, rst of all, we have to check if this additional assumption grants the property of $I(1)$ process to TFP and IST. For the validity, the cointegrating vector has to satisfy a speci c condition. It is helpful to refer to IST given in Eq.(11) and aggregate TFP in Eq.(13). Both logged TFP and IST are a special linear combination of logged sectoral productivities, $\ln Z_{c;t}$ and $\ln Z_{e;t}$, with di erent scale vectors; respectively, (; 1) and (1; 1). Now suppose that the uncovered cointegrating vector of $(\ln Z_{c;t}; \ln Z_{e;t})$ is (1;). To ensure the non-stationary property of TFP and IST, should not be equal to (1)= or 1. Accordingly, if the cointegrating vector of sectoral productivities satis es the conditions mentioned above, the non-stationarity of TFP and IST are preserved and Proposition 3 follows:

Proposition 3. Suppose $\ln A_t$, $\ln Q_t$, $\ln Z_{c;t}$ and $\ln Z_{e;t}$ follow $I(1)$ processes. Then, $\ln A_t$ and $\ln Q_t$ are cointegrated if and only if $\ln Z_{c;t}$ and $\ln Z_{e;t}$ are cointegrated.

Proof: refer to Appendix A

As we have already seen in Tables 2 and 3, Proposition 3 stands on the support of empirical ndings. Consequently, an appropriate model for a two-sector economy is better to introduce the cointegrated relation of sectoral productivities. In the following section, the cointegrated sectoral productivities are incorporated into a two-sector DSGE model and are used to estimate deep parameters and analyze the role of the stochastic common trend of sectoral productivities.

3 Model

Throughout Section 2, I have explained why we consider the cointegrated relationship of sectoral productivities in a two-sector economy model. Considering Proposition 3, this section develops a two-sector business cycle model extended from Ireland and Schuh (2008); their model is established for two-sector economy of consumption goods and equipment with both level and growth rate shocks of preference and productivities. The main di erence of this model is the cointegrated relationship of sectoral productivities. Additionally, to ensure fully mobile capital across sector, capital accumulation is allowed only at the aggregate level. Also, as real rigidities, capital adjustment cost and habit persistence in consumption are employed. Solving the competitive equilibrium, I introduce IST explicitly into the model; Ireland and Schuh (2008) regard IST as a shadow price.

3.1 The Household

Consider that the in nitely lived representative household has the preference, described over the habit persistent consumption, C_t, and hours worked, H_t, which is given by

$$E_0 \sum_{t=0}^{\infty} {}^{t} \{ \ln (C_t \quad C_{t-1}) \quad H_t / X_t \} ; \tag{16}$$

where and $\in [0; 1)$, respectively, denote the subjective discount factor and the degree of habit persistence. X_t stands for the preference shock. The preference shock consists of two stochastic components: level-stationary cyclical part, $X_{l;t}$, and growth-stationary trend part, $X_{g;t}$. The functional form of preference shocks are given by

$$X_t = X_{l;t} X_{g;t} ; \tag{17}$$

$$\ln X_{l;t} = {}_{xl} \ln X_{l;t-1} + {}_{xl;t} ; \tag{18}$$

$$\ln \frac{X_{g;t} / X_{g;t-1}}{{}_{xg}} = {}_{xg} \ln \frac{X_{g;t-1} / X_{g;t-2}}{{}_{xg}} + {}_{xg;t} ; \tag{19}$$

where ${}_j \in [0; 1)$ and ${}_j$, respectively, indicate the autoregressive coe cients and disturbance of stochastic process which is iid normal with mean zero and variance ${}^2_j$ for $j \in \{xl; xg\}$. ^{xg} stands

3.2 Firms

Two producing firms represent this model economy; one produces consumption goods and the other produces equipment. For the sake of clarity, I assume that all consumption goods are non-durables and all equipment are durables. This assumption is consistent with the definition that I used to construct the data of two-sector productivity in Section 2.1. Equipment is usually demanded for the two purposes: durable consumption and investment. By assuming all consumption goods are non-durable, however, I justify that all products of the equipment sector are used for investment without being spent for consumption. This assumption is by no means at odds; if we consider a household production, the durable consumptions can be regarded as an investment for the household's production. This assumption is also applied to the construction of observed data for consumption and investment.

Each firm $i \in \{c, e\}$, uses physical capital, $K_{i,t}$, and hours worked, $H_{i,t}$, as inputs to produce its output, $Y_{i,t}$, through a Cobb-Douglas type production function of homogeneous-degree-one as

$$Y_{c,t} = A_{c,t} K_{c,t}^{\alpha_c} (Z_{c,t} H_{c,t})^{1-\alpha_c}; \tag{26}$$

$$Y_{e,t} = A_{e,t} K_{e,t}^{\alpha_e} (Z_{e,t} H_{e,t})^{1-\alpha_e}; \tag{27}$$

where α_i denotes the substitute elasticity of physical capital for the production in sector i. $A_{i,t}$ indicates a Hicks-neutral productivity level shock of sector i and is assumed independent across sectors; these productivity level shocks are supposed to have mutually uncorrelated $AR(1)$ processes as follows:

$$\ln A_{c,t} = \rho_{ac} \ln A_{c,t-1} + \varepsilon_{ac,t} \tag{28}$$

$$\ln A_{e,t} = \rho_{ae} \ln A_{e,t-1} + \varepsilon_{ac,t}; \tag{29}$$

where $\rho_j \in [0, 1)$ and $\varepsilon_{j,t}$ denotes the autoregressive coefficient and disturbance term which is iid normal with mean zero and variance σ_j^2, for $j \in \{ac, ae\}$, respectively.

$Z_{i,t}$ is the productivity growth rate shock and exhibited as labor-augmented type. Following

Proposition 3, I assume that $Z_{c;t}$ and $Z_{e;t}$ are cointegrated and incorporated into the system through the vector error correction model (VECM) including the smooth transition non-linear error correction (STR NEC) as

$$\begin{bmatrix} \ln \frac{Z_{c;t}=Z_{c;t-1}}{zc} \\ \ln \frac{Z_{e;t}=Z_{e;t-1}}{ze} \end{bmatrix} = \begin{bmatrix} {}_{ce} & {}_{cc} \\ {}_{ee} & {}_{ec} \end{bmatrix} \begin{bmatrix} \ln \frac{Z_{c;t-1}=Z_{c;t-2}}{zc} \\ \ln \frac{Z_{e;t-1}=Z_{e;t-2}}{ze} \end{bmatrix} + \begin{bmatrix} f_c(ect_{t-1}) \\ f_e(ect_{t-1}) \end{bmatrix} + \begin{bmatrix} D_{ce} & D_{cc} \\ D_{ee} & D_{ec} \end{bmatrix} \begin{bmatrix} \end{bmatrix}$$

Eq.(26), and Eq.(27). Accordingly, these firms' profit-maximizing conditions imply that IST is the ratio of the marginal product of capital in equipment to the marginal product of capital in consumption-goods sector, which is given as follows:

$$Q_t = \frac{{}_{e}Y_{e,t}/K_{e,t}}{{}_{c}Y_{c,t}/K_{c,t}}. \tag{37}$$

3.3 Market Clearing

On the equilibrium, the four markets, consumption goods, equipment, capital and labor, of the model economy have to be cleared. Hence, the following market clearing conditions should be satisfied:

$$C_t = Y_{c,t}; \tag{38}$$

$$I_t = Y_{e,t}; \tag{39}$$

$$K_t = K_{c,t} + K_{e,t}; \tag{40}$$

$$H_t = H_{c,t} + H_{c,t}. \tag{41}$$

In addition, the aggregate output measured by unit of consumption goods is defined as

$$Y_t = Y_{c,t} + Y_{e,t}/Q_t. \tag{42}$$

3.4 Solution

The variables of this model economy possess non-stationary properties granted by Z_c, Z_e and X_g of $I(1)$ stochastic processes. Consequently, I need to transform each non-stationary variable into a stationary one on the balanced growth path. Since each variable grows with different rates along the balanced growth path, the functional form of the transformation depends on each of them. Through the following transformation equations, each non-stationary variable, denoted in upper-case, is replaced by its stationary form, denoted in lower-case, :$Y_t = y_t T^c_{t-1}$; $C_t = c_t T^c_{t-1}$; $H_t = h_t T^h_{t-1}$; ${}_{1,t} = {}_{1,t}/T^c_{t-1}$; ${}_{2,t} = {}_{2,t}/T^i_{t-1}$; $\tilde{R}_t = \tilde{r}_t T^c_{t-1}/T^i_{t-1}$; $\tilde{W}_t = \tilde{w}_t T^c_{t-1}/T^h_{t-1}$; $Q_t = q_t T^i_{t-1}/T^c_{t-1}$;

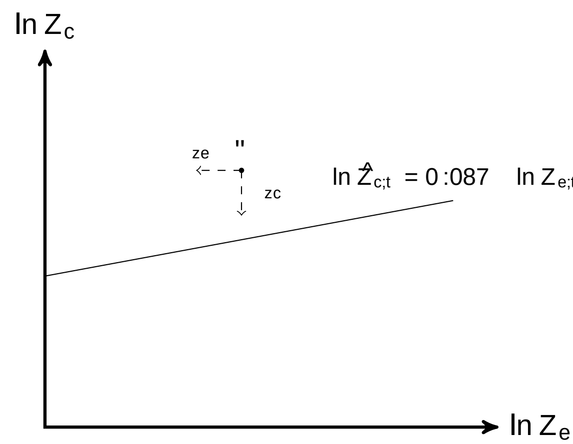
$K_t = k_t T^i_{t\ \ 1}$; $I_t = i_t T^i_{t\ \ 1}$; $Y_{c;t} = y_{c;t} T^c_{t\ \ 1}$; $Y_{e;t} = y_{e;t} T^i_{t\ \ 1}$; $K_{c;t} = k_{c;t} T^i_{t\ \ 1}$; $K_{e;t} = k_{e;t} T^i_{t\ \ 1}$; $H_{c;t} = h_{c;t} T^h_{t\ \ 1}$; $H_{e;t} = h_{e;t} T^h_{t\ \ 1}$; $X_{l;t} = x_{l;t}$; $A_{c;t} = a_{c;t}$; $A_{e;t} = a_{e;t}$, where $T^c_t = Z_{c;t}^{1\ \ c} Z_{e;t}^{\ c} X_{g;t}$, $T^i_t = Z_{e;t} X_{g;t}$ and $T^h_t = X_{g;t}$.

Table 4: Cointegrated relation of sectoral productivities

	TFP.cons	TFP.equip
Cointegration Vector	1	-0.087
Adjustment parameter	-0.653	-0.613

Notes: The estimated cointegrating vector and adjustment parameters are obtained by Johansen test for the dataset named `db6' represented in Table 2 and 3. The cointegrating vector is normalized by TFP.cons. TFP.cons and TFP.equip stand for the productivity of consumption goods and equipment, respectively.

Figure 1: Linear adjustment of the cointegrated sectoral productivities



estimated adjustment-speed vector is di erent to that of the fastest adjustment-speed vector. We can readily notice from Figure 1 that the linear adjustment from the deviation may not lead it back on the long-run equilibrium, if the deviation point, ", is far enough from the long-run equilibrium path.

How can we then ensure the global stability of the system of equations? One possible answer is suggested by Kapetanio, Shin and Snell (2006), who develop a method of testing non-linear cointegration using non-linear error correction. To check the applicability of their model, I test the non-linear cointegrated relationship of the annual sectoral productivities constructed from the EU KLEMS database using the methods of Kapetanio, Shin and Snell (2006). The statistic of F_{nec} tests the null hypothesis of no cointegration with no underlying assumptions. The statistic of F_{nec} tests the null hypothesis of no cointegration with the assumption that the switching point is zero.

Table 5: Cointegration test under non-linear error correction assumptions

	Case	Lags(AIC)	Test statistic	Critical value(95%)	Null hypothesis
F_{nec}	Constant	3	0.908	13.73	Accept
	Trend	3	1.112	16.13	Accept
F_{nec}	Constant	3	1.459	12.17	Accept
	Trend	3	1.873	15.07	Accept
t_{nec}	Constant	3	-3.224	-3.22	Reject
	Trend	3	-4.477	-3.59	Reject

Notes: The statistics of F_{nec} tests the null hypothesis of no cointegration with no under-lying assumptions. The statistics of F_{nec} tests the null hypothesis of no cointegration with the assumption that the switching point is zero. The statistic of t_{nec} tests the null hypothesis of no cointegration with the assumption that the switching point is zero and the error correction term follow unit roots process in the middle regime.

The statistic of t_{nec} tests the null hypothesis of no cointegration with the assumption that the switching point is zero and the error correction term follows the unit roots process in the middle regime. Table 5 shows the test statistics. The test statistics without underlying assumption (F_{nec}) and with the assumption of zero switching point (F_{nec}) fail to reject the null hypothesis of no cointegration. The test statistics with the assumption of zero switching point and the unit roots process in the middle regime t($_{nec}$), however, signi cantly reject the null hypothesis of no cointegration.

As such, the non-linear cointegrated relationship betw21(rell)-421(h)28(yp)-2s050 toregressive coe cients and the variance of disturbances. As in Ireland and Schuh (2008), and

Table 6: The maximum likelihood estimates and standard errors of the structural parameters

Parameter	Estimate	Standard error
	0.2028	0.0327
	0.3148	0.0410
	0.9349	0.0186
	-0.0551	0.0900
$_{c}$	0.3307	0.0310
$_{e}$	0.4009	0.0723
$_{cc}$	0.2986	0.1450
$_{ce}$	0.0000	0.0429
$_{ec}$	0.0000	0.0686
$_{ee}$	0.0000	0.0352
$_{c}$	-0.1825	0.4684
$_{e}$	1.7946	0.0671
D_{ce}	0.3000	0.0778
D_{ec}	0.0236	0.0949
$_{xl}$	0.8911	0.1324
$_{xg}$	0.5493	0.1156
$_{ac}$	0.0000	0.1141
$_{ae}$	0.0000	0.0702
$_{xl}$	0.0033	0.0014
$_{xg}$	0.0046	0.0009
$_{ac}$	0.0029	0.0005
$_{ae}$	0.0086	0.0020
$_{c}$	0.0042	0.0011
$_{e}$	0.0200	0.0052
$_{c}$	0.0004	0.0003
$_{i}$	0.0078	0.0000
$_{h}$	0.0023	0.0002

Notes: Sample period is 1948:Q2 to 2011:Q4. The observables are the growth rates of consumption, investment, and hours worked. Each of the observables is assumed to possess measurement error. During estimation $= 0.99$ and $= 0.025$ are imposed. The diagonal elements of VECM innovations, D_{cc} and D_{ee}, are normalized to unity.

the estimated 27 parameters estimated with standard errors, which come from a parametric bootstrapping procedure as in Ireland and Schuh (2008). I generate 1,000 sets of artificial data from the estimated model by assigning random disturbances for each period having the same length of actual data. The artificially generated 1,000 sets of data are used to estimate 1,000 sample parameters. The reported standard errors in Table 6 are the standard deviations of the samples. The model estimates a significant habit-persistence parameter, , of 0.2028; it is much higher than the estimate of Ireland and Schuh (2008) but a little bit lower than that of Schmitt-Grohé and Uribe (2011). The capital adjustment-cost parameter is estimated as 0.3148, which is even lower than reported in existing literature; however, the estimate is significant. The estimation allows the existence of measurement errors in consumption, investment, and hours worked series: denoted $_c$, $_i$, and $_h$, respectively. I curb the estimates of these measurement errors not to exceed 25% of the standard error of each series.

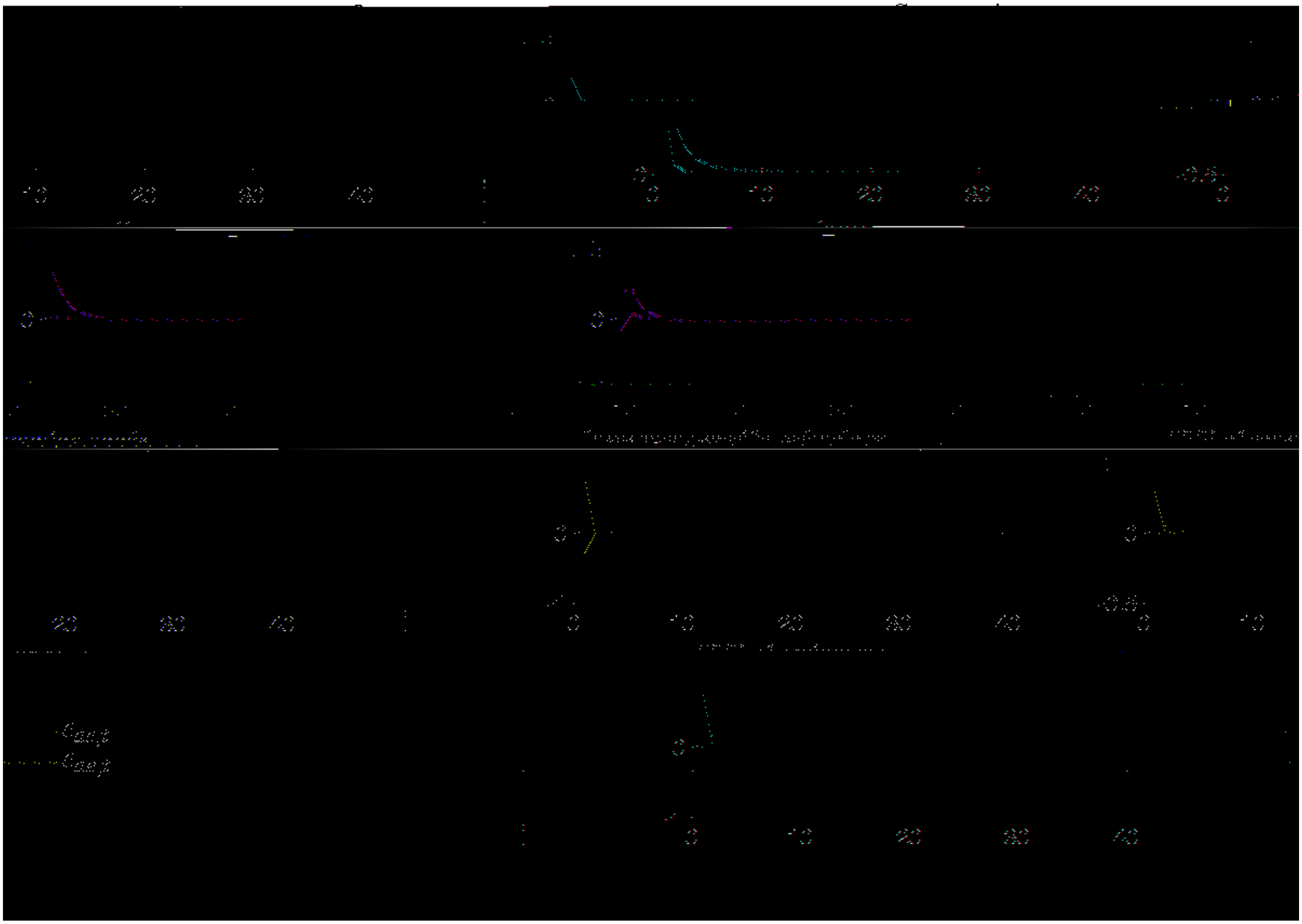
In the estimation, I estimate the capital share of each sector without assuming symmetry across sectoral production functions; most of the two-sector models, including Ireland and Schuh (2008), employ symmetric capital shares. The symmetry assumption, however, does not reflect the reality, but is done for convenience. The maximum likelihood method estimates the capital share of consumption goods, $_c$, as 0.3307 and that of equipment, $_e$, as 0.4009: the estimate of capital share in equipment production, however, has a twice as large standard deviation than that for consumption. The estimated sectoral capital shares are worth comparing with others: Ireland and Schuh (2008) estimate the capital share of 0.39 with s.e. 0.06, and Schmitt-Grohé and Uribe (2011) estimate 0.37 with s.e. 0.03. Therefore, we can see the estimate is not much different to the estimates of existing studies but rather lie within their two-standard error confidence intervals both in consumption goods and equipment. Additionally, the estimates correspond to the conventional wisdom, which says consumption goods production is relatively labor-intensive, meanwhile equipment production is capital-intensive.

The most interesting features of the estimation is the parameters of cointegration, volatility, and persistence of the shocks. The existence of cointegration can be tested by evaluating the estimate of .[12] If = 0, the error-correction term of non-linear VECM will vanish; it implies a regular VAR model. Applying the standard deviation of estimated , we can easily test the null hypothesis of = 0: we can reject the null because the estimated of 0.9349 lies far outside the two-standard deviation from the null. Accordingly, the cointegration of sectoral productivities is

[12] The maximum likelihood estimates have asymptotically normal distributions. Therefore, for hypothesis tests, we can apply t-test. See Canova (2007), pp. 225-228, for details.

con rmed. The persistence parameters of common trend shocks ($_{cc}$, $_{ce}$, $_{ec}$, and $_{ee}$) are estimated as 0.2986 and zeros, respectively, which mean the persistence of common trend shocks is delivered to the next period only through the consumption goods channel. The correlation parameters of the innovation of common trend, D_{ce} and D_{ec} indicate that the innovations of common trend shocks

Figure 4: Impulse responses on productivity shocks in level



Notes: Each panel shows the percentage deviation of output, consumption, investment, hours worked, IST, TFP of consumption goods sector, and TFP of equipment sector to a one-standard-deviation shock to the productivity level of each sector.

the preference shocks are not related to the changes in productivities, they have no e ect on sectoral TFPs.

Another notable fact in Figure 2 is the decrease of IST in the short run, which recovers its original state in the long run. This fact con rms Oulton (2007)'s argument: The relative price of

Table 7: Forecast-error variance decomposition

Quaters ahead	xl	xg	ac	ae	c	e
			Consumption			
1	14.0	52.5	26.2	2.2	1.3	3.8
4	4.2	49.5	2.5	0.4	20.0	23.4
8	2.5	48.6	0.9	0.2	20.1	27.6
12	1.9	47.3	0.5	0.2	17.7	32.5
20	1.1	44.9	0.2	0.1	13.7	40.0
40	0.5	41.7	0.1	0.0	9.4	48.3
			Investment			
1	10.1	1.0	0.1	84.0	0.2	4.6
4	6.8	10.9	0.0	11.3	0.0	70.9
8	3.7	15.8	0.0	3.8	0.0	76.7
12	2.5	17.6	0.0	2.4	0.0	77.4
20	1.6	18.9	0.0	1.5	0.0	78.0
40	0.9	19.4	0.0	0.9	0.0	78.8
			Hours worked			
1	41.9	43.8	2.0	11.1	0.2	1.0
4	16.5	71.4	0.2	2.2	0.1	9.5
8	8.6	78.1	0.1	0.8	0.0	12.5
12	5.8	81.9	0.0	0.5	0.0	11.7
20	3.6	86.5	0.0	0.3	0.0	9.5
40	2.0	91.8	0.0	0.2	0.0	6.0

Notes: The decomposed forecast error variances in consumption, investment, and hours worked are exhibited. The decomposition consists of the contribution of all 6 shocks to the forecast error variances.

equipment can change without the relative change of sectoral productivities. In the model economy, equipment production is capital-intensive, meanwhile consumption production is labor-intensive; these are estimated rather than assumed. The positive preference shocks increase labor supply and subsequently push down equilibrium wage. Accordingly, the production of consumption, which is labor-intensive, rise and it is accompanied by a decrease in consumption prices. Therefore, IST is decreasing in the short run. As we can see, however, the magnitude of the e ect is very limited. Consequently, we can say that Oulton's argument is right but not likely to be a dominant e ect in a real economy.

According to Figure 3, the shocks to common stochastic trend generally have persistent e ects on the model but the propagation paths di er for each source of shocks. The shock due to $_{e;t}$ has a very sizeable e ect on output, consumption, and investment. In particular, the e ect on investment is much larger than that on consumption and remains for a long period of time. $_{e;t}$ also increases the hours worked in the short run and shrink rapidly to its original level. The shock due to $_{c;t}$ mostly a ects the productivity of consumption goods. The e ect of $_{c;t}$ on the productivity of the equipment is negligibly small; subsequently, IST decreases almost permanently. However, investment does not shrink from that; instead it remains almost unchanged. The consequent e ect of $_{c;t}$

6 Conclusion

This paper theoretically and empirically presents the existence of a cointegrated relationship in sectoral productivities, which is motivated by the ndings of Schmitt-Grohe and Uribe (2011). Furthermore, I incorporate the cointegrated relationship of sectoral productivities into the two-sector model of Ireland and Schuh (2008). By introducing non-linear error correction into the model economy, I conduct maximum likelihood eo

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Appendix

A Proofs

A.1 Proof for Proposition 2

Suppose $\ln A_t$ and $\ln Q_t$ are cointegrated, then there exist $(1, \ )$ such that $\ln A_t + \ \ln Q_t = S_t^1$, where S_t^1

Suppose $\ln Z_{c;t}$ and $\ln Z_{e;t}$ consist of random walk, $_{c;t}$ and $_{e;t}$, and stationary parts, $e_{c;t}$ and $e_{e;t}$, as follows:

$$\ln Z_{c;t} = \quad _{c;t} + e_{c;t}$$

$$\ln Z_{e;t} = \quad _{e;t} + e_{e;t};$$

then $\ln A_t$ and $\ln Q_t$ are represented as follows:

$$\begin{aligned} \ln A_t &= \quad \ln Z_{c;t} + (1 \quad) \ln Z_{e;t} \\ &= \quad _{c;t} + (1 \quad) \; _{e;t} + \; e_{c;t} + (1 \quad) e_{e;t} \\ \ln Q_t &= \ln Z_{e;t} \quad \ln Z_{c;t} \\ &= \quad _{e;t} \quad _{c;t} + e_{e;t} \quad e_{c;t}: \end{aligned}$$

Since $\ln A_t$ and $\ln Q_t$ are cointegrated, there exists $(1 \quad)$ such that $\ln A_t + \quad \ln Q_t = S_t$ where S_t is a stationary process. $\ln A_t + \quad \ln Q_t$ can be rewritten as

$$\begin{aligned} \ln A_t + \quad \ln Q_t &= \quad _{c;t} + (1 \quad) \; _{e;t} + \quad _{e;t} \quad _{c;t} + D \\ &= (\quad) \; _{c;t} + (1 \quad + \quad) \; _{e;t} + D; \end{aligned}$$

where D is a stationary process, de ned as $e_{c;t} + (1 \quad) e_{e;t} + \; e_{e;t} \quad e_{c;t}$. Suppose further that $_{c;t}$ and $_{e;t}$ are not cointegrated, then the cointegrated $\ln A_t$ and $\ln Q_t$ requires the following conditions:

$$= 0; \text{ and}$$

$$1 \quad + \quad = 0:$$

The two equations, however, cannot be solved simultaneously. Therefore, $_{c;t}$ and $_{e;t}$ have to be cointegrated, which further implies the cointegration of $\ln Z_{c;t}$ and $\ln Z_{e;t}$.

Case2: $\ln Z_{c;t}$ and $\ln Z_{e;t}$ are cointegrated $\Rightarrow$ $\ln A_t$ and $\ln Q_t$ are cointegrated.

The Firms' Conditions

$$y_{c;t} = a_{c;t}(k_{c;t})^{\;c}(\;^{zc}_{t}h_{c;t})^{1\quad c} \qquad (B.1.7)$$

$$y_{e;t} = a_{e;t}(k_{e;t})^{\;e}(\;^{ze}_{t}h_{e;t})^{1\quad e} \qquad (B.1.8)$$

$$\tilde{r}_t = \quad _c y_{c;t}=k_{c;t} \qquad (B.1.9)$$

$$\tilde{w}_t = (1 \qquad _c) y_{c;t}=h_{c;t} \qquad (B.1.10)$$

$$q_t = \frac{_e y_{e;t}=k_{e;t}}{_c y_{c;t}=k_{c;t}} \qquad (B.1.11)$$

Market Clearing Conditions

$$k_t = k_{c;t} + k_{e;t} \qquad (B.1.12)$$

$$h_t = h_{c;t} + h_{e;t} \qquad (B.1.13)$$

$$c_t = y_{c;t} \qquad (B.1.14)$$

$$i_t = y_{e;t} \qquad (B.1.15)$$

$$y_t = y_{c;t} + y_{e;t}=q_t \qquad (B.1.16)$$

Growth Rates

$$^{c}_{t} = (\;^{zc}_{t})^{1\quad c}(\;^{ze}_{t})^{\;c}\;^{xg}_{t} \qquad (B.1.17)$$

$$^{i}_{t} =$$

Observable Variables

$$ {}^{C}_{t} = {}^{c}_{t\ 1}\frac{c_t}{c_{t\ 1}} \quad (B.1.20)$$

$$ {}^{I}_{t} = {}^{i}_{t\ 1}\frac{i_t}{i_{t\ 1}} \quad (B.1.21)$$

$$ {}^{H}_{t} = {}^{h}_{t\ 1}\frac{h_t}{h_{t\ 1}} \quad (B.1.22)$$

Exogenous Stochastic Processes

$$ ect_t \quad ect_{t\ 1} = \ln {}^{zc}_{t} \quad \ln {}^{ze}_{t} \quad (B.1.23)$$

$$ \begin{bmatrix} \ln ({}^{zc}_{t} = {}^{zc}) \\ \ln ({}^{ze}_{t} = {}^{ze}) \end{bmatrix} = \begin{bmatrix} {}_{cc} & {}_{ce} \\ {}_{ec} & {}_{ee} \end{bmatrix} \begin{bmatrix} \ln \ {}^{zc}_{t\ 1} = {}^{zc} \\ \ln \ {}^{ze}_{t\ 1} = {}^{ze} \end{bmatrix} + \begin{bmatrix} f_c(ect_{t\ 1}) \\ f_e(ect_{t\ 1}) \end{bmatrix} + \begin{bmatrix} D_{cc} & D_{ce} \\ D_{ec} & D_{ee} \end{bmatrix} \begin{bmatrix} {}_{zc;t} \\ {}_{ze;t} \end{bmatrix} \quad (B.1.24)$$

$$ \ln x_{I;t} = {}_{x;I} \ln x_{I;t\ 1} + {}_{xI;t} \quad (B.1.25)$$

$$ \ln({}^{xg}_{t} = {}^{xg}) = {}_{xg} \ln({}^{xg}_{t\ 1} = {}^{xg}) + {}_{xg;t} \quad (B.1.26)$$

$$ \ln a_{c;t} = {}_{ac} \ln a_{c;t\ 1} + {}_{ac;t} \quad (B.1.27)$$

$$ \ln a_{e;t} = {}_{ae} \ln a_{e;t\ 1} + {}_{ae;t} \quad (B.1.28)$$

B.2 The steady states

The steady-state values of the variables in the model economy are determined by exogenously given parameter set, , and the long-run average of the deterministic growth rates: ^{zc}, ^{ze} and ^{xg}. Substituting these parameters and growth rates into the Eqs.(B.1.17)-(B.1.19), we can get the steady-state of endogenous growth rates:

$$ {}^{c} = ({}^{zc})^{1\ \ c}({}^{ze})^{\ c\ \ xg} \quad (B.2.1)$$

$$ {}^{i} = {}^{ze\ \ xg} \quad (B.2.2)$$

$$ {}^{h} = {}^{xg}: \quad (B.2.3)$$

Using Eqs.(B.1.20)-(B.1.22), additionally, the long-run growth rate of the non-stationary variables are obtained as follows: $^{C} = \ ^{c}$, $^{I} = \ ^{i}$, and $^{H} = \ ^{h}$.

The household's optimization conditions exhibited in Eqs.(B.1.1)-(B.1.6), respectively, implies the following conditions on steady states:

$$_{1}c = \ _{1}; \tag{B.2.4}$$

$$1 = \ ^{xg} = \ _{1}\tilde{w}; \tag{B.2.5}$$

$$_{1}=q = \ _{2}; \tag{B.2.6}$$

$$_{2} \ ^{i} = \ f \ _{1}\tilde{r} + \ _{2}(1 \ \ )g; \tag{B.2.7}$$

$$c + i=q = \tilde{w}h + \tilde{r}k; \tag{B.2.8}$$

$$i = \ _{i}k; \tag{B.2.9}$$

where $_{1} = \frac{c}{c}$ and $_{i} = \ ^{i} \ \ 1 + \ $. Also, Eqs.(B.2.6) and (B.2.7) indicates

$$\tilde{r}q = rq; \tag{B.2.10}$$

where $rq = \ ^{i} = \ \ 1 + \ $.

Market clearing conditions, Eqs.(B.1.12)-(B.1.16), give the important steady-state equalities, respectively, as follows:

$$k = k_c + k_e \tag{B.2.11}$$

$$h = h_c + h_e \tag{B.2.12}$$

$$c = y_c \tag{B.2.13}$$

$$i = y_e \tag{B.2.14}$$

$$y = y_c + y_e=q \tag{B.2.15}$$

By considering Eq.(35) with stationary transformation, Eqs.(B.2.9), (B.2.10), (B.2.11) and

(B.2.14), we can write the steady-state capital of each sector in terms of aggregate capital stock:

Suppose that we have an equation given as follows:

$$f(X_t) + g(Y_t) = 0; \qquad \text{(B.3.1)}$$

where X and Y are strictly positive variables. Using the identity $X = e^{\ln X}$, we can rewrite Eq.(B.3.1) as

$$f\left(e^{\ln X_t}\right) + g\left(e^{\ln Y_t}\right) = 0. \qquad \text{(B.3.2)}$$

Taking the first-order Taylor expansion for Eq.(B.3.2) with respect to $\ln X$ and $\ln Y$ around the steady-state values, $\ln X$ and $\ln Y$, we can have

$$f(X) + f'(X)(\ln X_t - \ln X) + g(Y) + g'(Y)(\ln Y_t - \ln Y) = 0. \qquad \text{(B.3.3)}$$

Using the identity of $f(X) + g(Y) = 0$ and letting $\hat{x} = \ln X - \ln X$ and $\hat{y} = \ln Y - \ln Y$, Eq.(B.3.3) is simplified as

$$f'(X)\hat{x} + g'(Y)\hat{y} = 0. \qquad \text{(B.3.4)}$$

This standard-method of log-linearization can be coded on *Matlab* as follows:

```
ff_lv = subs(ff, {xx}, {exp(xx)});
grad = jacobian(ff_lv,xx);,
```

where ff stands for the system of equation before log-linearized and xx indicates a set of variables in the system. In the first line, *Matlab*, using the identity of $X = e^{\ln X}$, substitutes xx to logged xx. And then, take derivatives with respect to logged xx on the second line. With the simple two-line code, we can linearize more complicated system of equations easily.

Through the above method, I linearize the non-linear system of equations, Eqs.(B.1.1)-(B.1.28) around their steady state values.

B.4 Solving the Model

Theorem [Generalized Schur Form]. Let A and B be $n \times n$ matrices. If there is a $z \in \mathbb{C}$ such that $|B - zA| \neq 0$, then there exist matrices Q, Z, S and T such that

1. Q and Z are Hermitian , i.e. $Q^H Q = QQ^H = I_n$ and similarly for Z, where H denotes the Hermitian transpose.

2. T and S upper triangular.

3. $QA = SZ^H$ and $QB = TZ^H$.

4. There is no i such that $s_{ii} = t_{ii} = 0$.

Moreover, the matrices Q, Z, S and T can be chosen in such a way as to make the diagonal entries s_{ii} and t_{ii} appear in any desired order.

For ordering of i, the ones satisfying $|s_{ii}| > |t_{ii}|$ will be chosen to appear first; these s_{ii} and t_{ii} pairs are called stable generalized eigenvalues.

written out as

$$S_{22}E_t[u_{t+1}] = T_{22}u_t.$$

If S_{22} and T_{22} constitute a (weakly) unstable matrix pair, $|s_{ii}| < |t_{ii}|$ (for weakly $|s_{ii}| \leq |t_{ii}|$), then any solution to Eq.(B.4.2) with bounded variance must satisfy $u_t = 0, \forall t$ (for weakly, unless $\;= 0$). Given $u_t = 0, \forall t$, the first block of Eq.(B.4.4) should hold

$$S_{11}E_t[s_{t+1}] = T_{11}s_t. \tag{B.4.6}$$

If S_{11} and T_{11} constitute a stable matrix pair, $|s_{ii}| > |t_{ii}|$, then S_{11} is invertible. Hence we may write

$$E_t[s_{t+1}] = S_{11}^{-1}T_{11}s_t. \tag{B.4.7}$$

Rewrite Eq.(B.4.5) as

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = Z \begin{bmatrix} s_t \\ u_t \end{bmatrix}$$

follows[14];

$$L(d|\;) = \quad \frac{TI}{2}\ln(2\;) \quad \frac{1}{2}\sum_{t=1}^{T}\ln|\;_{t|t\;1}| \quad \frac{1}{2}\sum_{t=1}^{T} "_t \;{}_{t|t\;1}^{\;1} "_t ; \qquad \text{(C.1.1)}$$

where T shows the time-length of the observed-vector d, and I is the number of element of vector d, and $"_t$ and $\;_{t|t\;1}$ indicate the one-period-ahead forecast error of the observed-vector and its mean-square error, respectively.

For the consistency purpose from the previous sections, I suppose the state-space of this model economy as follows:

$$x_{t+1} = Cx_t + v_{t+1};$$

$$d_t = Dx_t + w_t;$$

where x and d respectively represent the state-vector of $k \;\; 1$, and the observed-vector of $I \;\; 1$. v and w

mean-square error:

$$d_{t|t-1} = Dx_{t|t-1};$$

where K_t implies the Kalman-gain given by

$$K_t = C_{t|t-1} D' \Omega^{-1}_{t|t-1}. \tag{C.1.9}$$

D Evaluating the model: Variance decomposition[16]

This section ascertains how to decompose the forecast error variance for the observable variables, such as consumption, investment, and hours worked into percentage due to each of the model shocks.

We can rewrite the state space equation and decision rule as follows:

$$x_{t+1} = P x_t + v_{t+1}; \tag{D.1.1}$$

$$y_t = F x_t. \tag{D.1.2}$$

Eq.(D.1.1) can be rewritten as MA representation:

$$(1 - PL) x_t = v_t$$

$$x_t = \textstyle\sum_{j=0}^{\infty} P^j v_{t-j} \tag{D.1.3}$$

The s-period-ahead forecast error of state vector on the information of time t is

$$x_{t+s} - \hat{x}_{t+s|t} = \sum_{j=0}^{s-1} P^j v_{t+s-j}; \tag{D.1.4}$$

and MSE of the forecast is exhibited as

$$E[(x_{t+s} - \hat{x}_{t+s|t})(x_{t+s} - \hat{x}_{t+s|t})'] \equiv \Omega_{x;s} = \Sigma_v + P \Sigma_v P' + P^2 \Sigma_v P'^2 + \dots + P^{s-1} \Sigma_v P'^{s-1}. \tag{D.1.5}$$

[16] This section mostly comes from the technical appendix of Ireland and Schuh (2008). I just redefine some variables to fit to the model economy and try to increase the readability.

Next we can get the forecast error of the non-state vector of Eqs.(D.1.2) as

$$y_{t+s} - y_{t+s|t} = F(x_{t+s} - x_{t+s|t}). \qquad \text{(D.1.6)}$$

Then MSE of the forecast for non-state vector is

$$E[(y_{t+s} - y_{t+s|t})(y_{t+s} - y_{t+s|t})'] \equiv \Sigma_{y;s} = F\Sigma_{x;s}F'. \qquad \text{(D.1.7)}$$

What we are interested in this analysis is mainly on the behavior of non-stationary aggregate variable such as consumption, investment, and hours worked per worker. Accordingly, we would get the variance decomposition for these non-stationary variables. In what follows, I describe the procedure for the variance decomposition of consumption as an example.

From the model solution given above we can rewrite the decision rule for consumption growth rate as follows:

$$\ln C_t - \ln C_{t-1} - \ln g^c = F_{gc} x_t; \qquad \text{(D.1.8)}$$

where F_{gc} indicate the row for the consumption growth (g_t^c) in matrix F. Then we can derives the following s-period-ahead forecasts from Eq.(D.1.8):

$$\ln C_{t+s} - \ln C_t - s\ln g^c = F_{gc}\sum_{j=1}^{s} x_{t+j}; \qquad \text{(D.1.9)}$$

$$\ln C_{t+s|t} - \ln C_t - s\ln g^c = F_{gc}\sum_{j=1}^{s} x_{t+j|t}. \qquad \text{(D.1.10)}$$

Then the forecast error and MSE of forecast are derived as

$$\ln C_{t+s} - \ln C_{t+s|s} = F_{gc}\sum_{l=1}^{s}\left(x_{t+l} - x_{t+l|t}\right) = F_{gc}\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j} \qquad \text{(D.1.11)}$$

$$E\left[\left(\ln C_{t+s} - \ln C_{t+s|t}\right)\left(\ln C_{t+s} - \ln C_{t+s|t}\right)'\right] = F_{gc}E\left[\left(\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j}\right)\left(\sum_{j}^{s}\right.\right.$$

where $\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j}$ is extended as

$$\begin{aligned}
\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j} &= \sum_{l=1}^{s}\left\{ v_{t+l} + P v_{t+l-1} + \cdots + P^{l-1} v_{t+1} \right\} \\
&= [\{v_{t+1}\} \\
&\quad + \{v_{t+2} + P v_{t+1}\} + \cdots \\
&\quad + \{v_{t+s} + P v_{t+s-1} + \cdots + P^{s-1} v_{t+1}\}] \\
&= v_{t+s} + (I+P) v_{t+s-1} + \cdots + (I + P + \cdots + P^{s-1}) v_{t+1} .
\end{aligned}$$

Then the middle term of Eq.(D.1.12) is represented as

$$E\left[\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j}\right]\left[\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j}\right]' = \Sigma_v + (I+P)\Sigma_v (I+P)' + \cdots + (I+P+\cdots+P^{s-1})\Sigma_v (I+P+\cdots+P^{s-1})' . \tag{D.1.13}$$